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A stochastic multi-cluster model of freeway traffic

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Abstract. A stochastic approach based on the Master equation is proposed to describe the process of formation and growth of car clusters in traffic flow in analogy to usual aggregation phenomena such as the formation of liquid droplets in supersaturated vapour. By this method a coexistence of many clusters on a one-lane circular road has been investigated. Analytical equations have been derived for calculation of the stationary cluster distribution and related physical quantities of an infinitely large system of interacting cars. If the probability per time (or p) to decelerate a car without an obvious reason tends to zero in an infinitely large system, our multi-cluster model behaves essentially in the same way as a one-cluster model studied before. In particular, there are three different regimes of traffic flow (free jet of cars, coexisting phase of jams and isolated cars, highly viscous heavy traffic) and two phase transitions between them. At finite values of p the behaviour is qualitatively different, *i.e.*, there is no sharp phase transition between the free jet of cars and the coexisting phase. Nevertheless, a jump-like phase transition between the coexisting phase and the highly viscous heavy traffic takes place both at $p \to 0$ and at a finite p. Monte-Carlo simulations have been performed for finite roads showing a time evolution of the system into the stationary state. In distinction to the one-cluster model, a remarkable increasing of the average flux has been detected at certain densities of cars due to finite-size effects.

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1 Introduction

The formation and growth of clusters is a widely known phenomenon in physics. We mention a formation of liquid droplets in a supersaturated vapour [1]. The formation of car clusters (jams) at overcritical densities in traffic flow is an analogous phenomenon in sense that cars can be considered as interacting particles [2], and the clustering process can be described by similar equations. In particular, the probability that the system has a given cluster distribution at a time t in both cases can be described by the stochastic Master equation. The transition probabilities depend on the specific physical model under consideration. It should be noted that the traffic flow and spontaneous emergence of car clusters has been studied by different authors (see *e.g.* [3–8]) based on different models and approaches. In spite of the complexity of real traffic [9], we believe that some general features, such as spontaneous formation of jams and some general scaling properties of traffic flow [10,11] exist, which can be described and understood by relatively simple models. Especially, particle hopping models and, in particular, the cellular automaton model developed by Nagel and Schreckenberg [10,12–16] historically plays an important role in the development of traffic flow theory and in practical traffic engineering. The Nagel-Schreckenberg model is still of current interest and recently has been revisited [17]. The deterministic car following theory [18–20] also gives support to understanding of real traffic. Our purpose is to extend and improve the one-cluster stochastic model of one-lane circular road, proposed and developed in references [21–23], allowing a coexistence of many clusters. It has been shown and discussed in reference [22] that the proposed approach allows to describe and interpret the experimental traffic data reflected in the fundamental flux-density diagram. In this paper the main attention is paid to the qualitative changes in the behavior of interacting cars (in particular, the existence and the character of phase transitions) which can be observed in the actually considered multi-cluster model, as compared to the one-cluster model studied before.

2 The model

Here we consider a model of traffic flow on a one-lane circular road of length L. For convenience, the length

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L is chosen

$$L = M(\ell + \Delta x_{\rm clust}), \tag{1}$$

where ℓ is the effective length of a car and $\ell + \Delta x_{\rm clust}$ is the distance between the front bumpers of two neighbouring cars in a jam (car cluster), as defined in our previous work [22], and M is a natural number. The distance between the front bumpers of two neighbouring cars, in general, is $\ell + \Delta x$. The total number of cars N is eliminated by $N \le M$, where $N = M$ corresponds to the marginal case when the jam with car density $\varrho_{\rm clust} = 1/(\ell+\Delta x_{\rm clust})$ exists over the whole road. Similarly as in reference [22], we have used the optimal velocity approximation to describe the behaviour of individual drivers depending on the local density of cars or on headway Δx. The maximal velocity of each car is $v_{\max}$. The desired (optimal) velocity $v_{\rm opt}$, depending on the distance between two cars Δx, is given in dimensionless variables $w_{\rm opt} = v_{\rm opt}/v_{\max}$ and $\Delta y = \Delta x/\ell$ by the formula

$$w_{\rm opt}(\Delta y) = \frac{(\Delta y)^2}{d^2 + (\Delta y)^2}\,, \tag{2}$$

where the parameter $d = D/\ell$ is the interaction distance. D is the distance between two cars corresponding to the velocity value $v_{\max}/2$. In distinction to our previous model [21–23], where one car cluster existed at any time, here we consider a more realistic multi-cluster case where the total number of clusters of congested cars (jams)

$$N_{\rm cl} = \sum_{k=1}^{N} N_k \tag{3}$$

may be varied. N_k is the number of clusters of size k, *i.e.*, those consisting of k cars. In principle, we allow an absence of any congestion corresponding to $N_k = 0$ for all k. Some stochastic event or perturbation of the free traffic flow is necessary to initiate formation of a new cluster. Such stochastic events are simulated assuming that any car belonging to the free flow can reduce its velocity to $v_{\rm opt}(\Delta x_{\rm clust})$, *i.e.*, become a single congested car or a cluster of size $k = 1$. The probability of such an event per time for a given free car is w_+^*. A cluster of size 1 appears also when a two-car cluster is reduced by one car. In this case cluster with $k = 1$ is a car which still have not accelerated after this event. In any case, cluster of size 1 in our model is defined as a single car moving with the velocity $v_{\rm opt}(\Delta x_{\rm clust})$. In such a way, the total number n of congested cars is

$$n = \sum_{k=1}^{N} k N_k\,, \tag{4}$$

and the number of free cars is $n_{\rm free} = N - n$. According to our definition, the length of the cluster of size k is $\ell k + (k-1)\Delta x_{\rm clust}$, which means that the total length of the congested part of the road is

$$L_{\rm clust} = \ell n + (n - N_{\rm cl})\Delta x_{\rm clust}\,. \tag{5}$$

Thus, with account for equation (1), the average distance $\Delta x_{\rm free} = \ell \Delta y_{\rm free}$ between two cars outside the jam (or free cars) distributed over the free part of the road with length $L_{\rm free} = L - L_{\rm clust}$ is given by

$$\Delta y_{\rm free}(n, N_{\rm cl}) = \frac{M - N + (M - n + N_{\rm cl})\Delta y_{\rm clust}}{N - n + N_{\rm cl}} \tag{6}$$

where $\Delta y_{\rm clust} = \Delta x_{\rm clust}/\ell$. In the particular one-cluster case $N_{\rm cl} = 1$ equations (5) and (6) agree with corresponding formulae in reference [22].

The traffic flow is described as a stochastic process where adding a vehicle to a given car cluster (any of $N_{\rm cl}$ clusters) is characterized by a transition frequency (attachment probability per time unit) $w_+(n, N_{\rm cl})$ and the opposite process by a frequency $w_-(n, N_{\rm cl})$. The stochastic variables are N_k with $k = 1, 2, ..., N$, whereas the transition frequencies depend on n and $N_{\rm cl}$, as discussed below. We have assumed that the free cars are distributed uniformly over the spacings between clusters, *i.e.*, all these parts of the free road are characterised by the same mean headway $\Delta y_{\rm free}(n, N_{\rm cl})$ defined by equation (6), which allows us to use the ansatz for transition frequencies proposed in reference [22]. However, at $n_{\rm free} < N_{\rm cl}$ the ansatz for w_+ is corrected, taking into account that some of $N_{\rm cl}$ parts of the free road contain no cars. We have introduced the probability, represented by theta (step) function,

$$R(n, N_{\rm cl}) = 1 + (n_{\rm free}/N_{\rm cl} - 1)\,\theta(N_{\rm cl} - n_{\rm free}), \tag{7}$$

that at a given time moment there exists at least one car in the considered part of the free road, assuming that the distribution of free cars is maximally uniform. In such a way, our ansatz for the transition frequencies reads

$$w_+(n, N_{\rm cl}) = \frac{b}{\tau}\,\frac{w_{\rm opt}(\Delta y_{\rm free}(n, N_{\rm cl})) - w_{\rm opt}(\Delta y_{\rm clust})}{\Delta y_{\rm free}(n, N_{\rm cl}) - \Delta y_{\rm clust}} \times R(n, N_{\rm cl}) \tag{8}$$

$$w_-(n, N_{\rm cl}) = 1/\tau = \text{const.}\,, \tag{9}$$

where $b = v_{\max}\tau/\ell$ denotes a dimensionless parameter, and τ is a time constant which can be interpreted as the waiting time for the escape (detachment) of the first car out of the jam into free flow [22]. Equations (7) and (8) ensure that $w_+(N, N_{\rm cl}) = 0$, therefore n cannot become larger than N. We have excluded any merging and splitting of clusters which is usually also done to describe aggregation in supersaturated systems like droplets [1]. Because in our model $\Delta y_{\rm clust}$ is strictly constant, such processes are impossible due to purely geometrical aspects.

The stochastic variables N_k may be considered as components of an N-dimensional vector $\mathbf{N} = (N_1, N_2, \ldots, N_N)$ or $\mathbf{q} = \sum_k N_k \mathbf{q}_k$ where $\mathbf{q}_k$ is a unit vector the i-th component of which is $\delta_{i,k}$. In such a notation, the stochastic Master equation describing the evolution of the probability distribution function $P(\mathbf{q}, T)$ with the dimensionless time $T = t/\tau$ reads

$$\begin{aligned}
&\frac{1}{\tau}\frac{\mathrm{d}P(\mathbf{q},T)}{\mathrm{d}T} = (N-n+1)\, w_+^*\, P(\mathbf{q}-\mathbf{q}_1,T) \\
&+ \sum_{k=2}^{N} (N_{k-1}+1) w_+(n-1,N_{\mathrm{cl}}) P(\mathbf{q}+\mathbf{q}_{k-1}-\mathbf{q}_k,T) \\
&+ (N_1+1)\, w_-(n+1,N_{\mathrm{cl}}+1)\, P(\mathbf{q}+\mathbf{q}_1,T) \\
&+ \sum_{k=1}^{N-1} (N_{k+1}+1) w_-(n+1,N_{(\mathrm{cl})}) P(\mathbf{q}+\mathbf{q}_{k+1}-\mathbf{q}_k,T) \\
&- \left[(N-n)\, w_+^* + N_{\mathrm{cl}}\,(w_+(n,N_{\mathrm{cl}}) \right. \\
&+ \left. w_-(n,N_{\mathrm{cl}}))\right]\, P(\mathbf{q},T)\,. \qquad (10)
\end{aligned}$$

The stochastic process has several reaction channels, written on r.h.s. of equation (10), which are transitions changing the cluster distribution from $\mathbf{N} = (N_1,\ldots,N_{k-1},N_k,\ldots,N_N)$ to $\mathbf{N}' = (N_1+1,\ldots,N_{k-1}+1,N_k-1,\ldots,N_N)$ and *vice versa* (valid for $k > 2$). The formation and dissolution of a pre-cluster (jam of size $k=1$) and a dimer (jam of two vehicles) have also to be considered. Generally equation (10) describes condensation and evaporation of car clusters due to stochastic one-step processes as attachment or detachment of one car only.

3 Stationary cluster distribution for an infinitely large system

The stationary cluster distribution for an infinitely large system ($M \to \infty$, $N \to \infty$) is derived in this section based on the stationary solution $P(\mathbf{q}) = \lim_{T\to\infty} P(\mathbf{q},T)$ of the Master equation (10) corresponding to the condition

$$\frac{\mathrm{d}P(\mathbf{q},T)}{\mathrm{d}T} = 0. \qquad (11)$$

When the stationary cluster distribution is reached in an infinitely large system, the relative fluctuation of the number N_k of clusters of a given size k is negligible, *i.e.*, the density $C(k) = \langle N_k/M\rangle$ of clusters consisting of k cars is

$$C(k) = M^{-1} \sum_i i\, P_k(i) = N_k^*/M \qquad (12)$$

where N_k^* is the most probable value of N_k and $P_k(i) = M^{-1}\delta(i/M - N_k^*/M)$ is the stationary probability that there exists i clusters of size k. The function $P_k(i)$ can be obtained from $P(\mathbf{q})$ by a summation over all possible values of N_m ($\mathbf{q} = \sum_m N_m \mathbf{q}_m$) except the value of N_k which is fixed $N_k = i$. In such a way, on the basis of equations (10), (11), and (12) we obtain the following equations

$$\begin{aligned}
&\mathrm{d}C(0)/\mathrm{d}T = -p\, C(0) + C(1) = 0 \\
&\mathrm{d}C(k)/\mathrm{d}T = [Q + \delta_{k,1}(p-Q)]\, C(k-1) \\
&- (1+Q)\, C(k) + C(k+1) = 0 \quad : \; k \geq 1 \qquad (13)
\end{aligned}$$

where $C(0) = n_{\mathrm{free}}/M$, $p = \tau w_+^*$, and $Q = \tau w_+(n,N_{\mathrm{cl}})$. We call p the stochastic perturbation parameter. This is a probability per dimensionless time unit for a given free car to decelerate (become a single congested car) without an obvious reason. Q is constant at given values of p and c, where c is the dimensionless total density of cars defined by $c = N\ell/L = c_{\mathrm{clust}}\, N/M$ where $c_{\mathrm{clust}} = 1/(1+\Delta y_{\mathrm{clust}})$. An unambiguous relation

$$C(k) = C(0)\, p\, Q^{k-1} \qquad (14)$$

follows from equations (13) for $k \geq 1$. $Q < 1$ corresponds to a physical solution because at $Q \geq 1$ sums (3) and (4) diverge. Equation (14) is not valid at $k \sim M$ since the density function $C(k)$ is meaningful at large N_k only. In this aspect, the question arises about the existence of one or several clusters of size $k \sim M$. Let us assume that there exists such a cluster of size $k = \mu\, M$ with $\mu > 0$ at some time moment T. The time evolution of this cluster is described by an averaged equation

$$\mathrm{d}\,\langle\mu\rangle\,/\mathrm{d}T = M^{-1}\,[Q(T) - 1], \qquad (15)$$

where the averaging of number of cars joining and leaving the cluster is performed over time intervals much larger than τ, but small enough to ensure that the relative variation of $\langle\mu\rangle$ during a time interval is small. Since $Q < 1$ corresponds to the stationary cluster distribution, stable clusters with $\mu \neq 0$ at $M \to \infty$ cannot exist at $T \to \infty$, *i.e.*, they dissolve, as evident from equation (15).

The value of $C(0)$ in (14), as well as the relative part of the congested cars $r = n/N$ and the average cluster size $s = rN/N_{\mathrm{cl}}$ can be easily calculated from equations (3), (4) and (14) with account for the relation $N = n + n_{\mathrm{free}}$. This yields

$$C(0) = \frac{c}{c_{\mathrm{clust}}}\,\frac{(1-Q)^2}{p+(1-Q)^2}, \qquad (16)$$

$$r = \frac{p}{p+(1-Q)^2}, \qquad (17)$$

$$s = \frac{r\,(1-Q)}{p\,(1-r)} = \frac{1}{1-Q}\,. \qquad (18)$$

Since the smallest cluster size is 1, $s \geq 1$ always holds. It follows from equation (18) that $n_{\mathrm{free}}/N_{\mathrm{cl}} = (1-Q)/p$. This quantity is larger than 1 for reasonably small values of p considered here, which means that R in equation (7) is equal to 1 and transition probability equation (8) reduces to

$$Q = b\, \frac{w_{\mathrm{opt}}(\Delta y_{\mathrm{free}}) - w_{\mathrm{opt}}(\Delta y_{\mathrm{clust}})}{\Delta y_{\mathrm{free}} - \Delta y_{\mathrm{clust}}} \qquad (19)$$

with

$$\Delta y_{\mathrm{free}} = \frac{(1-c)s - cr(s-1)\Delta y_{\mathrm{clust}}}{cs - cr(s-1)}, \qquad (20)$$

as consistent with equation (6) and the definitions of c, r, and s. Equations (17) to (20) together with equation (2) can be solved numerically, and this represents the solution of our model in the thermodynamic limit depending on the total dimensionless density of cars c and the stochastic perturbation parameter p. In a real situation drivers stop or randomly decelerate their cars without an obvious reason very seldom, which means that $p = \tau w_+^*$ is small, and, therefore, the asymptotic solution at $p \to 0$ is of special interest. Although, stable car clusters with $k \sim M$ do not exist, behaviour of the system in the limit $\lim_{p\to 0} \lim_{M\to\infty}$ is similar as in the case of one-cluster model considered in references [21,22] where one macroscopic cluster of a size proportional to length of road L appeared at certain densities c. This is true since the average size of spontaneously appearing clusters diverges at $\lim_{p\to 0} \lim_{M\to\infty}$ in our model, too. The above limit means that the solution is found at $M \to \infty$ for any given p, which then is tended to zero. In this case $s \to \infty$ is the only physical solution at densities $c_1 < c < \min\{c_2, c_{\text{clust}}\}$ where

$$\begin{aligned} c_{1,2} &= 1/(1+\Delta y_{1,2}) \,, \\ \Delta y_{1,2} &= B \pm \sqrt{B^2 - d^2 + 2B\Delta y_{\text{clust}}}, \end{aligned} \tag{21}$$

with

$$B = \frac{b\,d^2}{2[d^2 + (\Delta y_{\text{clust}})^2]} \,\cdot \tag{22}$$

Note that $\Delta y_{1,2}$ are roots of the equation $Q = 1$ solved with respect to Δy_{free}. The discriminant is supposed to be positive, and this is the condition (*cf.* Eq. (18) in Ref. [22]) at which formation of large clusters is possible, in principle. Thus, the asymptotic solution, describing the situation where large clusters coexist with parts of road with free traffic, reads

$$\begin{aligned} s &\simeq 1/\sqrt{ap} \to \infty\,; & r &\simeq 1/(1+a) \\ Q &\simeq 1 - \sqrt{ap} \to 1\,; & \Delta y_{\text{free}} &\simeq \Delta y_1\,, \end{aligned} \tag{23}$$

where

$$a = \frac{1 - c(1+\Delta y_{\text{clust}})}{c(1+\Delta y_1) - 1} \,\cdot \tag{24}$$

Such an asymptotic solution exists within $c_1 < c < c_{\text{clust}}$, whereas at $c_1 < c < \min\{c_2, c_{\text{clust}}\}$ this is the only physical solution. c_1 is the critical density, the same as in reference [22], at which homogeneous traffic flow becomes unstable and large clusters emerge if c is increased. Note that a diverges if the critical density c_1 is approached from larger c values. From equations (21) and (24) we see that $1/a$ and, consequently, r tends to zero linearly. This means that the critical behaviour of the order parameter r is described by the critical exponent $\beta = 1$. A similar result has been obtained in the Nagel-Schreckenberg traffic flow model [15]. At $c < c_1$ the only physical solution at $\lim_{p\to 0} \lim_{M\to\infty}$ is that corresponding to the homogeneous traffic flow, *i.e.*,

$$\begin{aligned} Q &\simeq \frac{2B[c^{-1} - 1 + \Delta y_{\text{clust}}]}{d^2 + (c^{-1}-1)^2}\,; \quad s = \frac{1}{1-Q}\,; \\ r &\simeq p/(1-Q)^2 \to 0\,; \quad \Delta y_{\text{free}} \simeq c^{-1} - 1\,. \end{aligned} \tag{25}$$

Such a solution has a physical meaning at $Q < 1$ which holds not only at $c < c_1$, but also at $c_2 < c < c_{\text{clust}}$ if $c_2 < c_{\text{clust}}$. In this case there are two physically meaningful asymptotic solutions (23) and (25) within $c_2 < c < c_{\text{clust}}$. The question is which of these solutions represents the stable state of the system. Following an analogy with the one-cluster model, there should be a jump between the states described by these solutions at some density $c_2 < c_{\text{jump}} < c_{\text{clust}}$.

4 The fundamental diagram

One of the most important characteristics of traffic flow is the fundamental diagram showing the flux J of cars as function of the total density $\varrho = N/L$. According to the definition given in references [21,22], we have

$$J = \lim_{t\to\infty} \frac{1}{t} \int_0^t \varrho(x,t')v(x,t')\,\mathrm{d}t' \,, \tag{26}$$

where $\varrho(x,t')$ and $v(x,t')$ are local density and local velocity depending on coordinate x and time t'. Since equation (26) holds for any coordinate x, we can perform an averaging over x replacing $\varrho(x,t')v(x,t')$ by the average value

$$\begin{aligned} \langle \varrho(x,t')v(x,t')\rangle_x = {} & \frac{L_{\text{clust}}(t')}{L}\, v_{\text{opt}}(\Delta x_{\text{clust}})\, \varrho_{\text{clust}}(t') \\ & + \frac{L_{\text{free}}(t')}{L}\, v_{\text{opt}}(\Delta x_{\text{free}}(t'))\, \varrho_{\text{free}}(t') \end{aligned} \tag{27}$$

where $\varrho_{\text{clust}}(t') = n(t')/L_{\text{clust}}(t')$ is the average density of cars in clusters, and $\varrho_{\text{free}}(t') = (N - n(t'))/L_{\text{free}}(t')$ is the average density of cars over the free part of the road at a time moment t'. In this case $L_{\text{clust}}(t')$, $L_{\text{free}}(t')$, and $\Delta x_{\text{free}}(t') = \ell \Delta y_{\text{free}}(t')$ are determined according to equations (5) and (6) where n and N_{cl} are the current stochastic values of these quantities at a time moment t'. Taking into account these relations (as well as the definitions $c = \ell N/L$ and $r = n/N$), from (26) we obtain

$$\begin{aligned} j = b\,c \sum_{n,N_{\text{cl}}} & P(n,N_{\text{cl}})\, [r(n)\, w_{\text{opt}}(\Delta y_{\text{clust}}) \\ & + (1-r(n))\, w_{\text{opt}}(\Delta y_{\text{free}}(n,N_{\text{cl}}))] \end{aligned} \tag{28}$$

where $j = J\tau$ is the dimensionless flux, $b = \tau v_{\max}/\ell$, and $P(n,N_{\text{cl}})$ is the part of the total time during which the number of congested cars is n and the number of clusters is N_{cl}. Another interpretation of quantity $P(n,N_{\text{cl}})$ is the stationary probability to find the system in a state with values of the stochastic variables N_k corresponding to the given n and N_{cl}. In the thermodynamic limit $M \to \infty$ the probability $P(n,N_{\text{cl}})$ has a sharp δ-function like maximum at the most probable values of n and N_{cl}, therefore equation (28) reduces to

$$j(c) = b\,c\,[r\, w_{\text{opt}}(\Delta y_{\text{clust}}) + (1-r)\, w_{\text{opt}}(\Delta y_{\text{free}})] \tag{29}$$

where quantities r and Δy_{free} are given by equations (17) and (20). In the limit $\lim_{p\to 0}\lim_{M\to\infty}$ the flux

$$j(c) = \frac{bc(1-c)^2}{(cd)^2 + (1-c)^2} \tag{30}$$

corresponds to the solution (25) without congestions, and

$$j(c) = 1 - c + c(bw_{\text{opt}}(\Delta y_{\text{clust}}) - \Delta y_{\text{clust}}) \tag{31}$$

holds for the cluster-phase solution given by equation (23). These equations and their illustration in the fundamental diagram (see Fig. 5) agree with the previously obtained results for the one-cluster model [21,22] and represent an exact analytical solution of our model in the considered limit. The question is merely about the density c_{jump} at which the jump in $j(c)$ from one solution to the other solution should take place.

5 Results and discussion

Here we discuss the results for an infinite system obtained on the basis of equations derived in previous sections and compare them with the results of Monte-Carlo (MC) simulation of stochastic trajectories for finite systems. The same set of dimensionless control parameters $b = 8.5$, $d = 13/6$, and $\Delta y_{\text{clust}} = 1/6$ has been used in all calculations. These values have been found in reference [22] by matching the theory with experimental data from German highways [9]. The stationary value of r (the relative part of congested cars) depending on the dimensionless density c is shown in Figure 1 (the upper picture). Similar results for $1/s$ (s is the average cluster size) also are shown here (the lower picture). In this figure the analytical solutions in the limit $\lim_{p\to 0}\lim_{M\to\infty}$ (first the limit $M \to \infty$ is found at a given $p > 0$), are shown by thick solid lines. The obtained analytical results in this limit clearly shows the existence of three different regimes of traffic flow, like in the one-cluster model [21,22], *i.e.*, free flow of cars at small densities ($c < c_1$), congested traffic or coexisting (cluster) phase at intermediate densities ($c_1 < c < c_{\text{jump}}$), and highly viscous overcrowded homogeneous state at high densities ($c_{\text{jump}} < c < c_{\text{clust}}$).

It is evident that there is a breakpoint at the first critical density $c = c_1$, and spontaneous formation of infinitely large clusters takes place at $c > c_1$. This phenomenon can be understood in analogy with the formation of liquid droplets in supersaturated vapour [1]. The singularity appears only if $p \to 0$, whereas at finite values of p there is no sharp phase transition in vicinity of $c = c_1$, which means that in this case our multi-cluster model behaves in a qualitatively different way as compared to the one-cluster model in references [21,22]. This can be seen from Figure 1 where solutions at $p = 0.001$ and $M \to \infty$ are shown by thin solid lines. It is interesting to note that a qualitatively similar result has been obtained in the cellular automaton model, *i.e.*, a sharp phase transition is not observed if some stochasticity is present [16]. Note that at a finite p the average cluster size is finite at any densities of cars. In this aspect, the proposed multi-cluster model provides some realistic description of the car distribution over a very long road with many jams, *i.e.*, in distinction to the one-cluster model, it allows to predict the average number of clusters and the average cluster size in a congested traffic. It is evident that there are two solutions at large densities: that with larger values of r and s corresponds to the cluster phase, whereas another one reflects the homogeneous state. Physically, in both cases there is a large average car density, but the distinguishing feature of the cluster phase is the existence of blanks in the dense traffic which in our model are interpreted as fragments of free phase coexisting with car clusters. The dense homogeneous state is described as a state without large

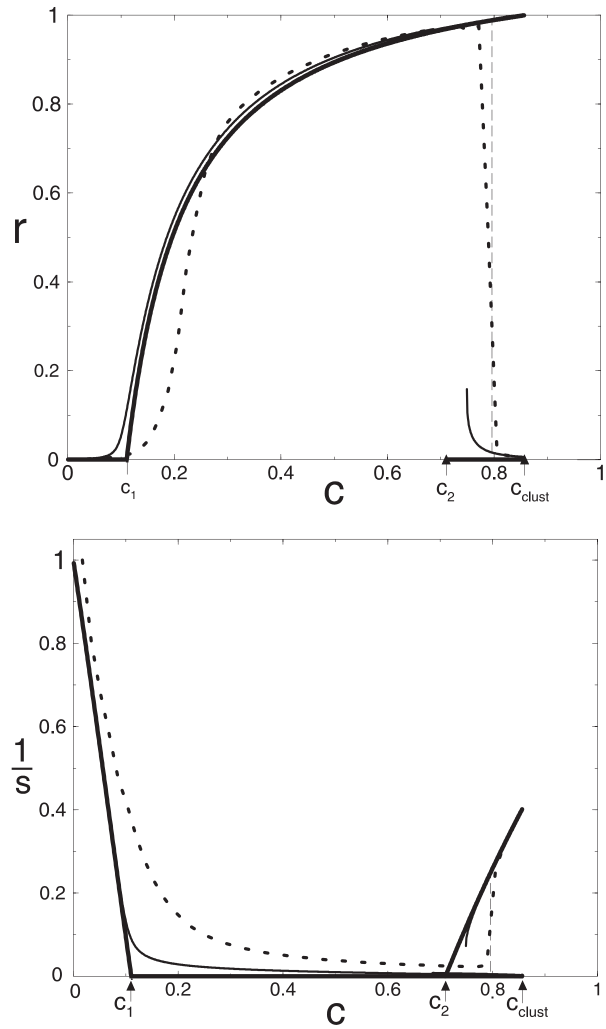

Fig. 1. The relative part of congested cars r (the upper picture) and the inverse value of the average cluster size $1/s$ (the lower picture) *vs.* dimensionless density c at different sizes M of the system and different values of the stochastic perturbation parameter p. Thin solid line: $p = 0.001$, $M = \infty$; thick solid line: $p = +0$, $M = \infty$; dotted line: MC simulation for $p = 0.001$, $M = 50$; the vertical dashed line indicates c_{jump}.

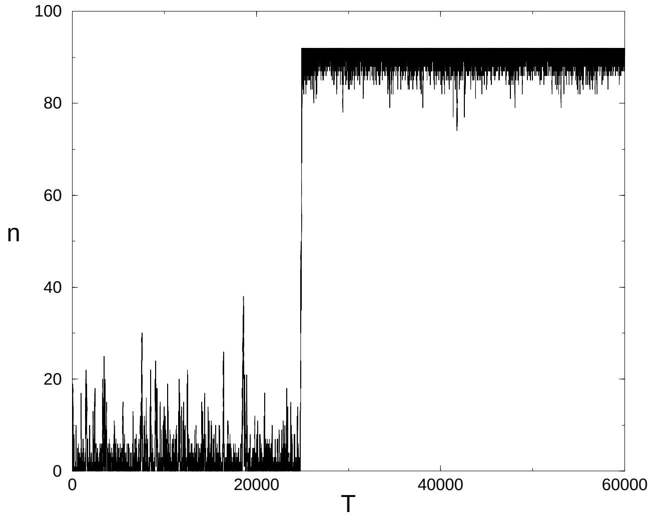

Fig. 2. A stochastic trajectory showing the total number of congested cars n *vs.* dimensionless time T at the stochastic perturbation parameter $p = 0.001$. The size of the system $M = 100$, and the total number of cars $N = 92$.

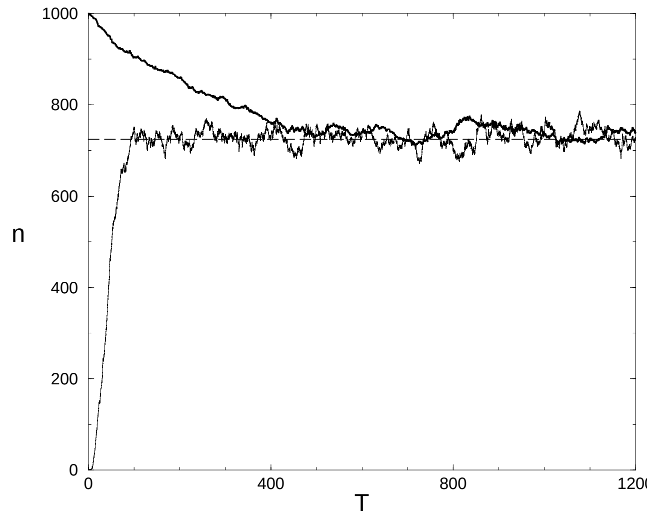

Fig. 3. Stochastic trajectories showing the total number of congested cars n *vs.* dimensionless time T starting with free flow (the thin line) and with large cluster (the thick line) at $p = 0.001$, $M = 3\,000$, and $N = 1\,000$. The horizontal dashed line denotes the theoretical average value 724.5 predicted from calculations at $M \to \infty$.

clusters or large-size inhomogeneities, but not as a single cluster which spreads over the whole road. Two solutions are present both at $p \to 0$ and at a finite p, which means that even at finite p values a jump-like (first-order) phase transition between two different states of the system takes place. It is suitable to consider the stationary probability $P^*(n)$ that the total number of congested cars is n. The true stationary state of the system at $T \to \infty$ is reflected by that solution which corresponds to the absolute maximum (located at $n \simeq rN$) of $P^*(n)$. We consider the limit $\lim\limits_{M\to\infty} \lim\limits_{T\to\infty}$, *i.e.*, first the stationary probability distribution is found at a given M. Physically it means that the considered time always is large enough to ensure that the system reaches the most stable state. Following an analogy with the one-cluster model [21,22], $\ln P^*(rN)$ for both solutions has the same value at some density c_{jump} where the jump-like phase transition takes place. We do not have an exact and rigorous result for c_{jump} in our model. However, we believe that the value $c_{\text{jump}} \simeq 0.796$ extracted from the one-cluster model and shown in Figure 1 by vertical dashed lines represents a reasonable estimate for our multi–cluster model. The existence of jump at $c \simeq 0.796$ is confirmed by the results of MC simulation shown in the figure by dotted line. The simulation results refer to a finite-size system with $M = 50$ and $p = 0.001$ and reflects the values of r and $1/s$ obtained by an averaging of r and s over a time interval $T = 200\,000$ to $500\,000$ with the initial condition $n = N$ and $N_{\text{cl}} = 1$. There is a principal problem to obtain adequate stationary results in the vicinity of c_{jump} by MC simulation of remarkably larger systems, since the time necessary for reaching the stable state of the system (if one starts from the metastable state) is too large. In fact, during a very long time the system stays in that state (homogeneous or cluster-phase) from which the simulation is started, and switchings between the states occur very seldom. A specific stochastic trajectory, reflecting the time evolution of n, where a switching from homogeneous state to the cluster-phase state occurs in the system with $M = 100$ and $N = 92$ is shown in Figure 2.

We have simulated stochastic trajectories for a large system with $M = 3\,000$ and $N = 1\,000$ to show the time evolution of the system and convergence to the stationary state described by equations in Section 3. In Figure 3 the time evolution of the total number of congested cars n is shown by stochastic trajectories starting from two different initial conditions, *i.e.*, from free flow ($n = 0$, $N_{\text{cl}} = 0$) and from a total congestion ($n = N = 1\,000$, $N_{\text{cl}} = 1$). In Figure 4 the same results are shown for the number of clusters N_{cl}. It can be seen from Figures 3 and 4 that irrespective to the initial conditions the total number of congested cars n (Eq. (4)) relatively fastly, as compared to N_{cl}, converges to values oscillating around the average value $< n > = rN \simeq 724.5$ consistent with the theoretical prediction (17) for an infinitely large system shown in Figure 3 by dashed line. The same is true for the total number of clusters N_{cl} (Eq. (3)) with the only difference that the convergence is much slower and oscillations around the average value $\langle N_{\text{cl}} \rangle = rN/s \simeq 14.13$ (the dashed line in Fig. 4) are larger. These results for N_{cl} confirm our theoretical prediction based on equation (15) that large clusters of size $k \sim M$, where $M \to \infty$, dissolve and, finally, the distribution (14) over cluster sizes is reached.

It is usually accepted to represent the traffic flow data in the flux-density diagram or the fundamental diagram of traffic flow. In Figure 5 the fundamental diagram is shown for an infinitely large system. It has been demonstrated in reference [22] that the fundamental diagram represented by equations (30) and (31) with the actually used values of control parameters is in a good agreement

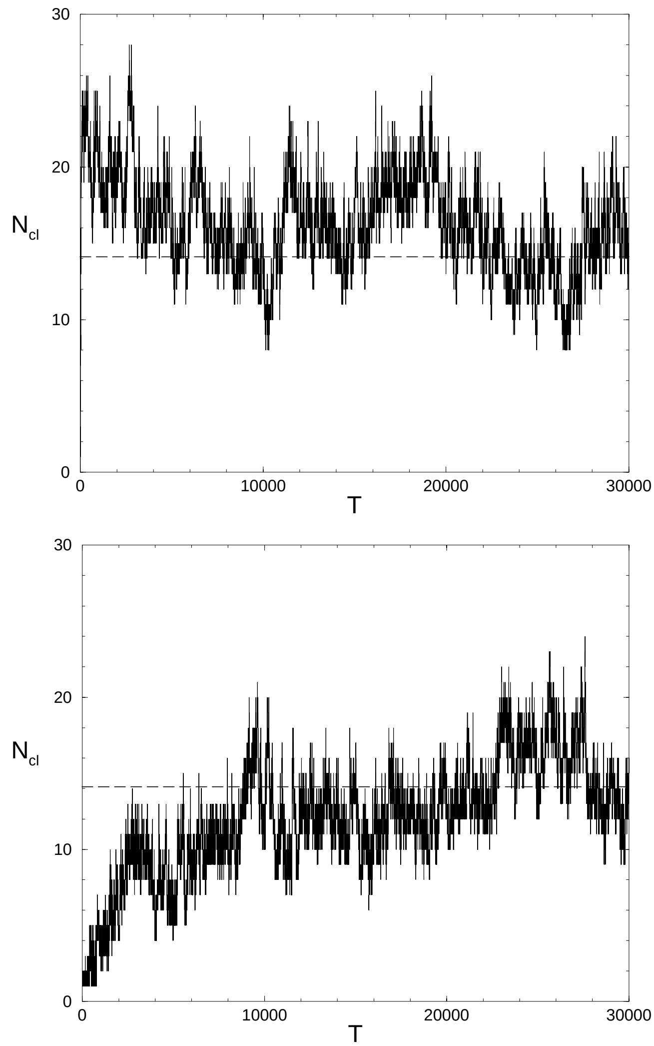

Fig. 4. Stochastic trajectories showing the total number of clusters N_{cl} *vs.* dimensionless time T starting with free flow (the upper picture) and with large cluster (the lower picture) at $p = 0.001$, $M = 3\,000$, and $N = 1\,000$. The horizontal dashed line denotes the theoretical average value 14.13 predicted from calculations at $M \to \infty$.

with experimental data. We have depicted this diagram, corresponding to the limit $\lim_{p\to 0} \lim_{M\to\infty}$, by thick solid lines.

At $c_2 < c < c_{\mathrm{clust}}$ there are two possible solutions, shown by two solid lines, one of which corresponds to the stable stationary state of the system. At $c_2 < c < c_{\mathrm{jump}}$ the stable state (the cluster phase) is reflected by the upper line, whereas at $c > c_{\mathrm{jump}}$ the stable state (highly viscous homogeneous phase) corresponds to the lower line. Our estimate for the critical density $c = c_{\mathrm{jump}}$ is indicated by an arrow. Continuations of the solutions (30) and (31) to the regions $c_1 < c < c_2$ and $0 < c < c_1$, respectively, are shown by dashed lines. We have depicted by thin solid line the solution at $p = 0.001$ and $M \to \infty$. In this case the jump at $c = c_{\mathrm{jump}}$ is so small that practically it is not seen. It is evident from the figure that the flux $j(c)$ represented by the thin solid line has a smooth maximum at $c \approx c_1$,

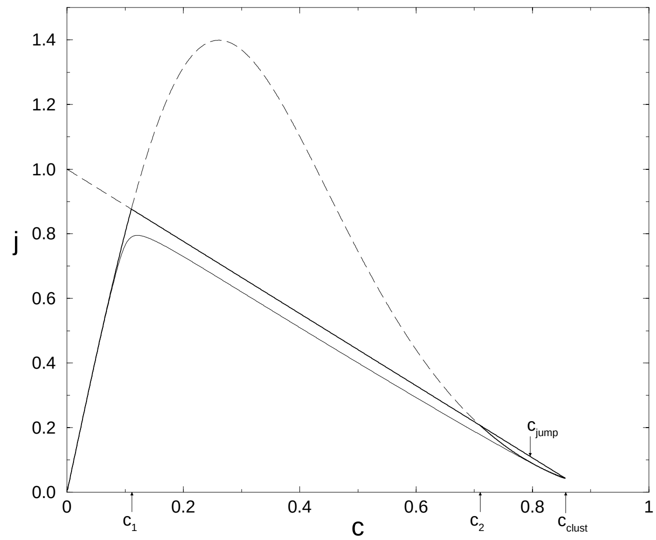

Fig. 5. The fundamental diagram of traffic flow (flux *vs.* density) for an infinitely long circular one-lane road at different values of the stochastic perturbation parameter p. Thick solid lines (continued by dashed lines): $p = +0$; thin solid line: $p = 0.001$; The jump between two different states of the system occurs at $c = c_{\mathrm{jump}}$ indicated by an arrow.

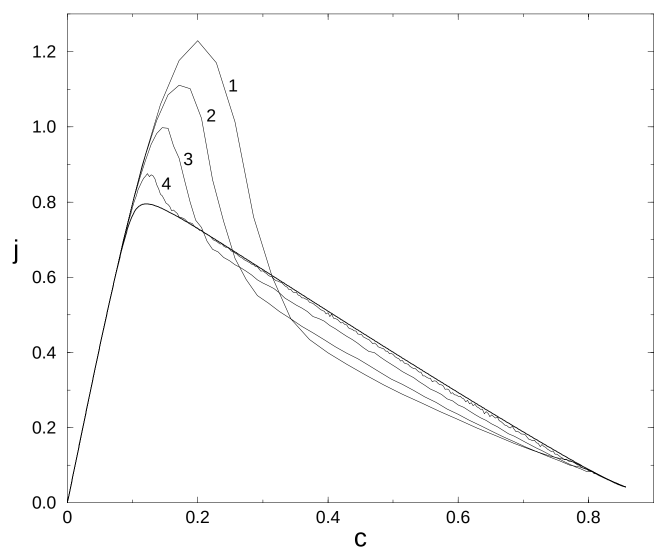

Fig. 6. The finite size effect on the flux-density diagram. Results of Monte-Carlo simulations averaged over a time interval $T = 200\,000$ to $500\,000$ at $p = 0.001$ and different sizes of the system: $M = 30$ (curve 1), $M = 50$ (curve 2), $M = 100$ (curve 3), and $M = 300$ (curve 4). The thicker smooth line, calculated from analytical equations, corresponds to $p = 0.001$, $M \to \infty$.

as consistent with the absence of sharp phase transition at $c = c_1$ for finite p values. At $p = +0$ three different regimes of traffic flow can be distinguished as discussed in the beginning of this section.

We have revealed and have shown in Figure 6 an interesting finite-size effect on the fundamental diagram, which has not been observed in the one-cluster model. In this figure the solution for an infinite system (29) at $p = 0.001$ is depicted by a smooth relatively thicker solid line. By thin solid lines, results of MC simulation are shown for finite roads of several lengths ($M = 30$, 50, 100, and 300 at $p = 0.001$) obtained by averaging (27) over a time interval $T = 200\,000$ to $500\,000$. As we see, in smaller systems the average flux is remarkably increased at densities slightly above c_1. We think that this is due to the stochastic fluctuations which are more important in finite systems as compared to the case $M \to \infty$ where the stationary cluster distribution (14) is determined and does not fluctuate. Remarkably larger values of the average flux for small systems at densities slightly above c_1 correlate with smaller values of r in this case, as evident from Figure 1 (the dotted line). It can be seen from Figure 6 that the flux-density diagram obtained by MC simulation converges to the result (29) for an infinite system, as M is increased.

6 Conclusions

In references [21–23] the stochastic theory of freeway traffic has been proposed and developed based on a single-cluster model for a one-lane circular road. Here we have extended this approach and have proposed and solved a more realistic model allowing the coexistence of many clusters.

Analytical equations have been derived for calculation of stationary cluster distribution and corresponding to this the average cluster size, the relative part of congested cars, and the stationary average flux for an infinitely large system of interacting cars. Completely analytical solutions have been obtained in the limit of an infinitely small value of the stochastic perturbation parameter p. It has been shown that spontaneous formation of large car clusters occurs at some critical density. The critical behaviour of the order parameter r (the relative part of congested cars) is described by the critical exponent $\beta = 1$, like in the Nagel-Schreckenberg cellular automaton model. At finite values of p, the behaviour of the system near the critical density c_1 is qualitatively different. In distinction to the case $p \to 0$, in this case there is no sharp phase transition with spontaneous formation of infinitely large clusters at $c = c_1$. The average cluster size always is finite. Nevertheless, even at finite p there is a jump-like phase transition from the cluster phase to the homogeneous state at large densities like in the one-cluster model studied before [21,22].

Monte-Carlo simulations have been performed for finite roads showing the time evolution (stochastic trajectories) of the system to the stationary state. The simulation results for large system ($M = 3\,000$) are in agreement with the analytical results for an infinite system. It has been shown that for small systems ($M = 30$, 50) the average flux of cars is remarkably increased at densities slightly above c_1 due to finite-size effects.

In conclusion, the developed multi-cluster model is helpful in understanding and interpretation of real traffic. In addition to the one-cluster model studied before, it allows to describe a real distribution of clusters (jams) of different sizes in a case of a long road with many jams.

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